The Guidance-Evasion Differential Game: Alternative Solvability, Relaxations, and Program Constructions

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***Abstract*—It is investigated differential game (DG) of guidance-evasion on a finite time interval. For Player I, the problem of guidance with target set under state constraints is considered. The goal of Player II is opposite. An analog of Krasovskii-Subbotin alternative is established. Relaxations of the initial DG are investigated. These relaxations correspond to weakening of game termination conditions for guidance problem. For investigation of DG, program iteration method is used.**

***Keywords*—differential game, program iteration method, relaxation.**

We consider differential game (DG) on a finite time interval. It is supposed that two sets-parameters are given: the target set for Player I and the set defining its state constraints. Player It is interested in guidance realization. The goal of Player II is opposite. For this DG, N.N. Krasovskii and A.I. Subbotin established the known alternative theorem. In this theorem, both sets were supposed closed in position space. In our investigation, an analog of the alternative theorem is obtained; in addition, with respect to set formative state constraints, it is supposed only closed-ness of sections. For the case when both sets-parameters of DG are closed in position space, relaxation of guidance problem is investigated. Namely, initial sets-parameters are replaced by neighborhoods selected with regard to some priority coefficient. We consider the choice of neighborhoods with extreme properties under condition of guaranteed solvability of the problem of Player I. As a result, a position function is realized. We investigate properties of this function. In particular, we set the fixed-point property and the cost function property for some auxiliary DG. In our constructions, known program iteration method (PIM) is used very widely. One variant of PIM [1] is used for construction of alternative partition of the set defining state constraints of Player I. The corresponding iterated procedure is realized in the set space. Another variant of PIM is analog of [2] and realized in function space. For first PIM variant, we consider systems of iterated procedures corresponding to replacement of initial sets-parameters by neighborhoods (see [3]). In addition, topological properties of program operators are established. These properties permit to obtain important statementsfor iterated procedure in function space.

It is important to note that under weakening of conditions for guidance game, we assume asymmetric variants with respect to target set and set defining state constraints. So, we can consider the setting with more severe constraints in guidance to target set. And conversely, the harder requirements with respect to validity of state constraints can be investigated. These variants are realized by the choice of a special priority parameter. So, we can consider different initial settings.

Now, we note base steps of our construction connected with investigation of DG of above-mentioned type.

1) The obtaining of a new variant of the alternative theorem for guidance-evasion DG. It is supposed that the set defining state constraints can be nonclosed in the position space but all sections of this set are closed. Namely, we have a graph of arbitrary closed-valued mapping. We consider non-anticipating strategies of Player I and special strategies with non-anticipating choice of correction times of Player II.

2) The construction of relaxations of guidance problem of Player I. The following construction is realized for closed DG (the target set and the set defining state constraints are closed in the position space). We consider DG with weakened conditions of termination for guidance problemby employment of neighborhoods for both sets-parameters. Sizes of neighborhoods are selected with employment of the priority coefficient. Therefore, we consider a scalar real parameter as analog of neighborhoods size supposing that this parameter is neighborhood size for the target set (the corresponding size for the set defining state constraints is realized by multiplying by the priority coefficient). Under fixed game position, we define the natural analog of the least size for which Player I guarantees guidance with the corresponding weakening of termination conditions (we keep in mind the replacement of sets-parameters by neighborhoods). Under enumeration of the game positions, we obtain a real-valued base function. Moreover, by employment of PIM realized for the set space, we define a sequence of functions convergent to our base function.

3) Construction of the new direct variant of PIM in function space realizing the above-mentioned sequence by iterated procedure. A new operator realizable in class of control-measures is proposed. With employ-

ment of this operator, an iterated procedure is realized. So, our base function is realized as limit of iteration sequences in a function space.

4) The fixed-point property of the base function. It is established that our base function is fixed point of operator from 3). And what is more, this base function has the property of order extremality in class of fixed points with natural properties. Finally, the order interval containing our base function is stated.

5) The property of a valued function. For every position the base function is a value function for a minimax-maximin DG with a special quality functional. In this game, Player I uses non-anticipating strategies and Player II uses special strategies with control of correction times. So, our base function defines a game value for every position in minimax-maximin game (two-personzero-sum game).

Now, we make more precise some elements of general setting. We fix two numbers t_0 and ϑ_0 for which t_0¡ϑ_0. Suppose that $T \triangleq [t_{0,}\vartheta_0]$ is a comprehending time interval (here and later, $\triangleq$ is equality by definition). We consider control processes on time intervals $[t_,\vartheta_0]$, $t \in T$. Let $\mathbb{R}^n$ be the phase space of our system, where $n \in \mathbb{N} \triangleq \{1;2;\ldots\}$; $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$ are nonempty compact sets, $p \in \mathbb{N}$ and q $\in \mathbb{N}$. In addition, P is the set of all instantaneous controls of Player I and Q is the set of all instantaneous controls of Player II. We fix a continuous function $f : T \times \mathbb{R}^n \times P \times Q \to \mathbb{R}^n$ defining control system

$$\dot{x} = f(t, x, u, v), \quad u \in P, \quad v \in Q. \quad (1)$$

Systems of such type are considered in DG theory; see [4,5] and other. Suppose that the system (1) satisfies the conditions of A.V. Kryazhimskii (see [6]); we keep in mind conditions of generalized uniqueness and uniform boundedness of programmed trajectories. We consider two sets M and N,

$$M \subset T \times \mathbb{R}^n \;\; and \; N \subset T \times \mathbb{R}^n. \quad (2)$$

Suppose that M is the target set for Player I and N is the set formative state constraints for this player. These constraints are realized by sections

$$Nt \triangleq \{x \in \mathbb{R}^n \mid (t, x) \in N\}, \; t \in T. \quad (3)$$

In Krasovskii-Subbotin alternative, the sets M and N were supposed to be close; see [5,6]. In our alternative variant, it is supposed that M is a closed set and all sections (3) are closed; the set itself N can be nonclosed in the position space

$$T \times \mathbb{R}^n = [t_0, \vartheta_0] \times \mathbb{R}^n \quad (4)$$

The guidance problem for Player I consists in the following: under given initial position (t_*, x_*) of the set (4), it is required to find a strategy U of Player I for which, under every trajectory $x_U \; (\cdot)$ starting from (t_*, x_*) and generating by U

$$\exists\, \vartheta \in [t_*, \vartheta_0] : ((\vartheta, \; x_U(\vartheta)) \in M) \\ \& \quad (t, \; x_U(t)) \in N \;\; \forall\, t \in [t_*, \vartheta]. \quad (5)$$

We consider (5) as a guidance realization. Then, our evasion problem consists in the following: for (t^*, x^*) of the set (4), it is required to find a strategy V of Player II for which, under every trajectory $x_V\,(\cdot)$ starting from (t^*, x^*) and generating by V, $\forall\, \vartheta \;\epsilon\; [t^*\;,\;\vartheta_0]$

$$((\vartheta, x_V(\vartheta))\, \epsilon M) \Rightarrow (\exists t \epsilon [t^*, \vartheta\, [: (t, x_V(t)) \notin N). \quad (6)$$

We consider (6) as an evasion realization. Of course, classes of strategies of Payers I and II must be coordinated. We consider the question about alternative solvability in the sense of (5) and (6). Moreover, we consider relaxation of guidance problem: namely, we consider an analog of (5) under replacement of M and N by neighborhoods with some agreement of neighborhood sizes. So, we use the next replacements

$$M \to M^{\varepsilon_1}, N \to N^{\varepsilon_2}, \quad (7)$$

where M^δ and N^δ are closed δ - neighborhoods of M and N (here and later, we suppose that M and N are closed sets in the position space), $\delta > 0$. In (7), we connect $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ by condition $\varepsilon_2 = \chi\, \varepsilon_1$, where $\chi > 0$, is a priority parameter. Of course, under these conditions, we strive to reduce ε_1; but, in addition, we require that Player I could guarantee the guidance realization under replacement (7). Then, we obtain the least value of ε_1 with the above-mentioned condition for every game position. In other words, we have an important position function with values

$$\varepsilon_0(t, x|\chi), (t, x) \in T \times \mathbb{R}^n. \quad (8)$$

It is established that given function (8) is lower semicontinuous; minorant and majorant of (8) are indicated. Moreover, we indicate a special operator Γon function space for which two important property of (8) are realized: 1) by Γ an iterated sequence with property of convergence to $\varepsilon_0(\cdot,\cdot|\chi)$ (8) is realized; 2) $\varepsilon_0(\cdot,\cdot|\chi)$ is a fixed point of Γ. Finally, under fixed position $(t_*, x_*)\,, \varepsilon_0(t_*, x_*|\chi)$ is DG price with a special cost functional (we keep in mind a zero-sum DG). So, a new variant of PIM is constructed; this is iterated procedure connected with Γ.

In article, rather new questions of DG theory are considered: a modification of alternative theorem, relaxation of the game guidance problem, and construction of a position function, realizing the value of relaxation parameter for every position. For these constructions, procedures of PIM are used. A new variant of PIM is proposed. The role of PIM is very essential. This method is an instrument of investigation of DG structure under very general conditions.

References

[1] A. G. Chentsov.The program iteration method in a game problem of guidance, Proc. Steklov Inst. Math. (Suppl.), 297, suppl. 1 (2017), 43–61.

[2] A. G. Chentsov.On the game problem of convergence at a given moment of time // Mathematics of the USSR-Izvestiya (1978), 12(2): 426-437.

[3] A.G.Chentsov, D.M.Khachay, Advanced Control Techniques in Complex Engineering Systems: Theory and Applications. 2019. P.129-161.- (Studies in Systems, Decision and Control; vol.203).

[4] N. N. Krasovskii and A. I. Subbotin, Positional Differential Games, Nauka, Moscow, 1974.

[5] Krasovskii N.N., Subbotin A.I., An alternative for the game problem of convergence. J. Appl. Math. Mech., 1970, vol. 34, no. 6, pp. 948-965. doi: 10.1016/0021-8928(70)90158-94.

[6] Kryazhimskii A.V. On the theory of positional differential games of approach-evasion. Soviet Math. Dokl., 1978, vol. 19, no. 2, pp. 408-412.