Transversality and Alternating Projections

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***Abstract*—Several kinds of 'regular' arrangement of a pair of sets near a point in their intersection are discussed: transversality, subtransversality and intrinsic transversality. Such regular intersection properties are crucial for the validity of qualification conditions in optimization as well as subdifferential, normal cone and coderivative calculus, and convergence analysis of computational algorithms. Dual characterizations of the properties in terms of Fréchet and limiting normals are provided.**

***Keywords*—transversality, subtransversality, intrinsic transversality, alternating projections, subdifferential, normal cone.**

Introduction

We continue the study of the 'regular' arrangement of a collection of sets near a point in their intersection [2-7]. Such *regular intersection* or, in other words, *transversality* properties are crucial for the validity of *qualification conditions* in optimization as well as subdifferential, normal cone and coderivative calculus, and convergence analysis of computational algorithms. Note also the well-known equivalences between transversality properties of collections of sets and the corresponding regularity properties of set-valued mappings. The origins of the concept of regular arrangement of sets in space can be traced back to that of *transversality* in differential geometry.

Alternating Projections

Assuming for simplicity that A and B are closed sets in finite dimensions, alternating projections are determined by a sequence (x_k) alternating between the sets:

$$x_{2k+1} \in P_B(x_{2k}),\ x_{2k+2} \in P_A(x_{2k+1})\ (k = 0, 1, \ldots),$$

with some initial point x_0; see Fig. 1 and 2. Here P_A and P_B stand for the (Euclidean) projection operators on the respective sets. This simple algorithm has a long history; see [5, 7]. It is often referred to as *von Neumann method* although some traces of this method can be found in the 19th century's publications.

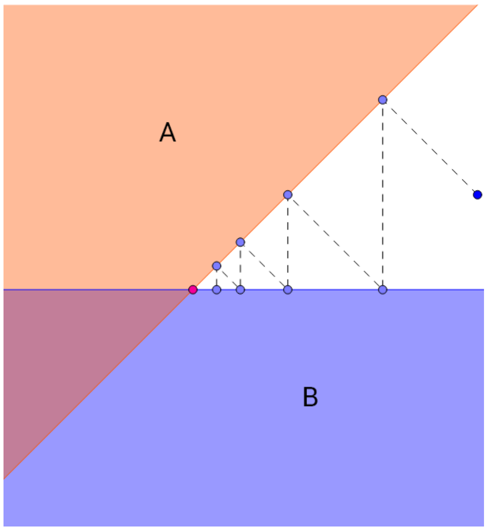

Fig. 1. Alternating projections

(Dolecki, 1982); (Ioffe, 1989); (local) *linear regularity* (Bauschke, Borwein, 1993); *linear estimate, linear coherence* (Penot, 1998, 2013); metric inequality (Ngai, Théra, 2001); subtransversality (Ioffe, 2015).

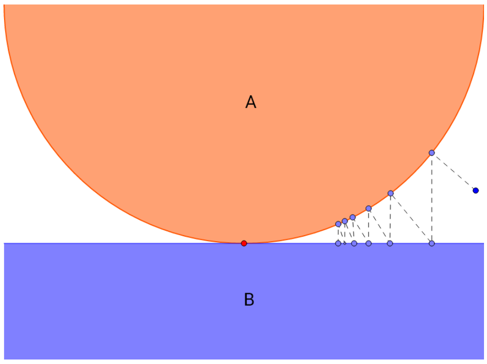

Fig. 2. Alternating projections

Subtransversality and Transversality

Let X be a normed vector space, $A, B \subset X$ and $\bar{x} \in A \cap B$.

Definition 1. The pair $\{A, B\}$ is *subtransversal* at $\bar{x}$ if there exist $\alpha, \delta > 0$ such that

$$\alpha d\,(x, A \cap B) \le \max\{d(x, A), d(x, B)\}$$

for all $x \in \mathbb{B}_\delta(\bar{x})$.

Definition 2. The pair $\{A, B\}$ is *transversal* at $\bar{x}$ if there exist $\alpha, \delta > 0$ such that

$$\alpha d\,(x, (A - x_1) \cap (B - x_2))$$
$$\le \max\{d(x, A - x_1), d(x, B - x_2)\}$$

for all $x \in \mathbb{B}_\delta(\bar{x})$ and $x_1, x_2 \in \delta\mathbb{B}$.

The next implication is straightforward:

Transversality $\Rightarrow$ Subtransversality

Theorem 1. Let X be Asplund and A, B be closed. The pair $\{A, B\}$ is transversal at $\bar{x}$ if and only if there exist $\alpha, \delta > 0$ such that $\|x_1^* + x_2^*\| > \alpha$ for all $a \in A \cap \mathbb{B}_\delta(\bar{x})$, $b \in B \cap \mathbb{B}_\delta(\bar{x})$, $x_1^* \in N_A(a)$ and $x_2^* \in N_B(b)$ satisfying $\|x_1^*\| + \|x_2^*\| = 1$.

Here $N_A(a)$ is the Fréchet normal cone to A at a.

Corollary 1. Let $\dim X < \infty$ and A, B be closed. The pair $\{A, B\}$ is *transversal* at $\bar{x}$ if and only if there exists $\alpha > 0$ such that $\|x_1^* + x_2^*\| > \alpha$ for all $x_1^* \in \bar{N}_A(\bar{x})$ and $x_2^* \in \bar{N}_B(\bar{x})$ satisfying $\|x_1^*\| + \|x_2^*\| = 1$.

Here $\bar{N}_A(\bar{x})$ is the limiting normal cone to A at $\bar{x}$.

Corollary 2. Let $\dim X < \infty$ and A, B be closed. The pair $\{A, B\}$ is transversal at $\bar{x}$ if and only if $\bar{N}_A(\bar{x}) \cap (-\bar{N}_B(\bar{x})) = \{0\}$.

(Mordukhovich, 1984); transversality (Clarke et al, 1998); normal qualification condition (Mordukhovich, 2006; Penot, 2013); *regular intersection* (Lewis & Malick, 2008); *linearly regular intersection* (Lewis et al, 2009); *alliedness* property (Penot, 2013); *transversal intersection* (Ioffe, 2015).

Theorem 2. Let $\dim X < \infty$ and A, B be closed and convex. Alternating projections converge linearly for any starting point close to $\bar{x}$ if and only if $\{A, B\}$ is *subtransversal* at $\bar{x}$.

(Bauschke, Borwein, 1993; Drusvyatskiy, Ioffe, Lewis, 2015; Luke, Teboulle, Thao, 2020).

In the nonconvex case, the situation is more complicated.

Definition 3 (Drusvyatskiy, Ioffe, Lewis, 2015 [1]). Let $\dim X < \infty$ and A, B be closed. The pair $\{A, B\}$ is *intrinsically transversal* at $\bar{x}$ if there exist $\alpha, \delta > 0$ such that

$$\max\left\{ d\left(\frac{b-a}{\|a-b\|}, N_A(a)\right), \; d\left(\frac{a-b}{\|a-b\|}, N_B(b)\right)\right\} > \alpha$$

for all $a \in (A \backslash B) \cap \mathbb{B}_\delta(\bar{x})$ and $b \in (B \backslash A) \cap \mathbb{B}_\delta(\bar{x})$.

The next implications are straightforward:

Transversality $\Rightarrow$ Intrinsic transversality $\Rightarrow$ Subtransversality

The following fact is of importance:

If A, B are convex, then Intrinsic transversality $\Leftrightarrow$ Subtransversality

Theorem 3 (Drusvyatskiy, Ioffe, Lewis, 2015 [1]). Let $dimX < \infty$ and A, B be closed. Alternating projections converge linearly for any starting point close to $\bar{x}$ if $\{A, B\}$ is *intrinsically transversal* at $\bar{x}$.

Intrinsic transversality has been extended to arbitrary normed linear spaces in [4]. Primal space characterizations of the property can be found in [8].

Acknowledgment

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References

[1] D. Drusvyatskiy, A. D. Ioffe, and A. S. Lewis, "Transversality and alternating projections for nonconvex sets," Found. Comput. Math., vol. 15, pp. 1637-1651, 2015.

[2] A. Y. Kruger, "Stationarity and regularity of set systems," Pacif. J. Optimiz., vol. 1, pp. 101-126, 2005.

[3] A. Y. Kruger, "About regularity of collections of sets," Set-Valued Anal., vol. 14, pp. 187-206, 2006..

[4] A. Y. Kruger, "About intrinsic transversality of pairs of sets," Set-Valued Anal., vol. 26, pp. 111-142, 2018.

[5] A. Y. Kruger and N. H. Thao, "Regularity of collections of sets and convergence of inexact alternating projections," J. Convex Anal., vol. 23, pp. 823-847, 2016.

[6] A. Y. Kruger, D. R. Luke and N. H. Thao, "About subtransversality of collections of sets," Set-Valued Var. Anal., vol. 25, pp. 701-729, 2017.

[7] A. Y. Kruger, D. R. Luke and N. H. Thao, "Set regularities and feasibility problems," Math. Program., Ser. B, vol. 168, pp. 279-311, 2018.

[8] N. H. Thao, T. H. Bui, N. D. Cuong, and M. Verhaegen "Some new characterizations of intrinsic transversality in Hilbert spaces," Set-Valued Var. Anal., vol. 28, pp. 5-39, 2020.