Variational Analysis: New Trends And Developments

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***Abstract*—We discuss some current trends of variational analysis related to second-order theory and meaningful applications. Among the most impressive ones from both viewpoints of variational theory and applications, there are optimal control problems for the so-called sweeping/Moreau processes described by discontinuous differential inclusions via the normal cones to moving sets. We formulate such problems and discuss their differences from conventional optimal control problems governed by differential equations and differential inclusions with Lipschitzian right-hand sides. The constructions of the method of discrete approximations married to advanced tools of second-order variational analysis allow us to establish the well-posedness and strong convergence of discrete approximations, obtain necessary optimality conditions for discrete counterparts of the controlled sweeping processes, and then to establish by passing to the limit those for local minimizers of the original sweeping control problems expressed entirely in terms of the given data. Some recent applications are also briefly discussed.**

***Keywords*—variational analysis, generalized differentiation, optimal control, differential inclusions, controlled sweeping processes, discrete approximations, necessary optimality conditions, generalized Euler-Lagrange conditions, Pontryagin's maximum principle.**

Variational analysis has been recognized as an active and fruitful area of mathematics that, on one hand, concerns the study of optimization-related problems and, on the other hand, applies optimization, perturbation, and approximation ideas to the analysis of a broad range of problems that may not be of a variational nature. This area can be considered as an outgrowth of the classical calculus of variations, optimal control theory, and mathematical programming, where the focus is on optimization of functions relative to various constraints and on the solution sensitivity with respect to perturbations. A characteristic feature of modern variational analysis is a broad involvement of objects with nonsmooth structures (nondifferentiable functions, sets with nonsmooth boundaries, and set-valued mappings), which appear naturally and frequently in the framework of optimization, equilibrium, and control problems while being often generated by the use of variational principles and techniques. Nonlinear systems and variational principles in applied sciences also give rise to nonsmooth structures and motivate the developments of new forms of analysis that rely on generalized differentiation. Starting with convex nonsmooth functions, subgradient mappings are inevitably set-valued and require tools of set-valued analysis for their study and applications. The last two decades have witnessed rapid progress in developing and applications of variational analysis and generalized differentiation. Starting with the fundamental monograph by Rockafellar and Wets [1] devoted to finite-dimensional aspects of variational analysis and their applications to constrained optimization, an enormous amount of publications on the subject appeared and great many applications were developed to various branches of mathematical sciences as well as to engineering, economics, biology, environmental and behavioral sciences, operations research, computer sciences, robotics, etc. We refer the reader to [1] and the author's monographs [2, 3], with the vast bibliographies and commentaries therein, for the genesis of ideas, basic constructions and results, and applications of variational analysis and generalized differentiation in finite and infinite dimensions. In particular, the second volume of [2] contains a rather comprehensive account of the methods and results of variational analysis applied to nonsmooth controlled systems governed by ordinary differential, functional differential, and partial differential equations and Lipschitzian differential inclusions with a variety of applications. New trends in variational analysis largely relates to the investigation and solving of meaningful problems, which often come from applications. A broad class of such problems are governed by discontinuous differential inclusions introduced originally by Moreau [4] in the form of the sweeping process

$$\dot{x}(t) \in -N\big(x(t);C(t)\big) \quad \text{a.e. } \ t \in [0,T] \quad \text{with } \ x(0) := x_0 \in C(0),$$

where $N(x;C)$ stands for the normal cone of convex analysis defined by

$$N(x;C) := \big\{v \in \mathbb{R}^n \big|\ \langle v, y-x\rangle \le 0,\ y \in C\big\} \quad \text{if } \ x \in C \ \text{ and } \ N(x;C) := \emptyset \text{ if } \ x \notin C$$

for the continuously moving convex set $C = C(t)$ at the point $x = x(t)$. The primal motivation of [4] mainly came from problems of elastoplasticity, but over the years the sweeping process and its modifications have been developed in dynamical system theory with many applications to various areas of mechanics, economics, traffic equilibria, robotics, etc.; see, e.g., the excellent recent survey [5] with the references therein. However, optimization and control problems for sweeping processes were formulated much latter. A primal reason for this situation is that the Cauchy problem for the sweeping process has a *unique* solution, and so there is nothing to optimize. First optimal control problems for sweeping processes were formulated with controls functions acting in additive perturbations as in [6]. Nevertheless, necessary optimality conditions for controlled sweeping processes were first established only in [7] (see also [10]) for a news class of problems with control functions acting in the moving set formalized as $C(t) = C(u(t))$ on $[0,T]$. Another type of controlled sweeping processes was introduced in [8], with deriving necessary optimality conditions, where control functions entered a linear ODE system adjacent to the sweeping dynamics. The very recent years have witnessed a rapidly growing interest to the derivation of necessary optimality conditions for various types of controlled sweeping processes with their broad applications to practical models; see, e.g., [11]– [20] and the references therein.

Here we pay the main attention to a general class of optimal control problems governed by a perturbed sweeping process over controlled polyhedral sets. Namely, we consider the optimal control problem (P) described as follows: minimize the cost functional of the Bolza type

$$J[x,a,b,u] := \varphi\big(x(T)\big) + \int_0^T \ell\left(t, x(t), a(t), b(t), u(t), \dot{x}(t), \dot{a}(t), \dot{b}(t)\right) dt$$

subject to the perturbed sweeping dynamics

$$\begin{cases} \dot{x}(t) \in -N\big(x(t); C(t)\big) + g\big(x(t), u(t)\big) \quad \text{a.e. } t \in [0,T], \\ x(0) = x_0 \in C(0) \subset \mathbb{R}^n \end{cases}$$

with trajectories $x(\cdot) \in W^{1,2}([0,T];\mathbb{R}^n)$ generated by measurable controls $u(\cdot) \in L^2([0,T];\mathbb{R}^d)$ in the additive perturbations of satisfying the constraint

$$u(t) \in U \subset \mathbb{R}^d \ \text{ a.e. } t \in [0,T]$$

as well as absolutely continuous controls $a(\cdot) = (a_1(\cdot), \ldots, a_m(\cdot)) \in W^{1,2}([0,T];\mathbb{R}^{mn})$ and $b(\cdot) = (b_1(\cdot), \ldots, b_m(\cdot)) \in W^{1,2}([0,T];\mathbb{R}^m)$ acting in the moving set

$$C(t) := \{x \in \mathbb{R}^n \mid \langle a_i(t), x\rangle \le b_i(t),\ i = 1, \ldots, m\}, \quad t \in [0,T],$$

under the the pointwise constraints on the entire time interval given by

$$\|a_i(t)\| = 1 \ \text{ for all } \ t \in [0,T],\ i = 1, \ldots, m,$$

$$\dot{a}_i(t) \in A_i \subset \mathbb{R}^n \ \text{ and } \ \dot{b}_i(t) \in B_i \subset \mathbb{R} \quad \text{a.e. } t \in [0,T],\ i = 1, \ldots, m,$$

where the initial points $x_0 \in C(0)$, $a_{i0} := a_i(0) \in \mathbb{R}^n$, $b_{i0} := b_i(0) \in \mathbb{R}$ are fixed together with the final time $T > 0$.

Observe that, besides the hard/pointwise constraints on control functions in both the additive perturbations $g(x,u)$ and the moving set $C(t)$ imposed above, we automatically have the mixed pointwise state-control constraints given by the bilinear inequalities

$$\langle a_i(t), x(t)\rangle \le b_i(t) \ \text{ for all } \ t \in [0,T] \ \text{ and } \ i = 1, \ldots, m,$$

which follow from the sweeping inclusion due to the normal cone definition ensuring that $x(t) \in C(t)$ on $[0,T]$. Together with the non-Lipschitzian (heavily discontinuous) sweeping dynamics in (P), this makes deriving necessary optimality conditions for local minimizers of (P) to be a very challenging task, which does not allow us to employ conventional techniques of the calculus of variations and optimal control theory.

It is important to emphasize that controlled sweeping differential inclusions are highly different from optimal control problems governed by smooth ODE systems as in the classical monograph by Pontryagin et al. [21] and by their nonsmooth counterparts (see, e.g., [22]), as well as by set-valued problems governed by Lipschitzian differential inclusions; see the books [2], [22] for more details and references. Note that the latter control systems play a prominent role in the differential game theory developed in the school by Krasovskii-Subbotin [23].

To advance the sweeping control theory, we employ the approach based on discrete approximations, which goes back to Euler in the classical calculus of variations. It was used by Pshenichnyi [24] to optimize Lipschitzian differential inclusions governed by set-valued mappings with convex graphs under rather restrictive assumptions. Those restrictions and the graph-convexity have been overcome by the author [3], [26], [27] for nonconvex Lipschitzian differential inclusions by using his version of variational analysis and generalized differentiation.

However, the situation with highly non-Lipschitzian sweeping differential inclusions is very different from Lipschitzian ones and requires much more involved techniques of variational analysis and discrete approximations. The realization of these techniques has been done in the aforementioned publications for particular cases of problem (P) formulated above. A crucial role in these developments is played by new methods and results of *second-order* variational analysis and generalized differentiation, since second-order generalized derivatives naturally appear in the adjoint systems to the sweeping dynamics due to its normal cone description.

In this way we construct well-posed sequence of discrete approximations of the sweeping control problem (P), establish strong convergence of discrete optimal solutions to the prescribed local minimizer of (P), derive necessary optimality conditions for solutions to discrete approximations, and finally obtain efficient necessary optimality conditions for local minimizers of the original problem (P) expressed entirely in terms of its initial data. The established optimality conditions include, in particular, the adjoint systems in the generalized Euler-Lagrange form based on the author's coderivative concept, and an appropriate extension of Pontryagin's maximum principle. The obtained results lead us to fruitful applications to practical models arising in mechanics, robotics, hysteresis, traffic equilibria, nanotechnology, etc.

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